Determinant

Determinant of order 2

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Determinant of order-3 (row operation)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - a_3b_2)$$

Column operation

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$

Q.
$$\begin{vmatrix} 3 & 1 & 2 \\ 4 & 0 & 1 \\ -1 & 2 & 3 \end{vmatrix} = 3(0-2) - 4(3-4) - 1(1-0)$$

$$= 3(-2) - 4(-1) - 1$$

$$= -6 + 4 - 1$$

$$= -3.$$

Properties of determinant

(1) The value of the determinant remain unchanged changing its rows (columns) into corresponding columns (rows).

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(2) If all the elements of any two rows (columns) of a determinant are identical then the value of the determinant vanishes or zero.

$$\begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = 0.$$

③ If all the elements of any two rows (columns) of a determi... are interchanged then the value of the determinant ch... its sign.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = K_1 \qquad \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix} = -K_1.$$

④ If all the elements of any rows (columns) of a determinan... are multiplied by a non-zero constant, then the value of ... determinant is also multiplied by the same Constant.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ Ka_2 & Kb_2 & Kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = K \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

⑤ If all the elements of any row (column) of a determinant are the sum of two terms, then the determinant can be expressed as the sum of two determinant.

$$\begin{vmatrix} a_1+p & b_1+q & c_1+r \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p & q & r \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

⑥ If all the element...
are increa... column) of determinant
... ...ed by a constant multiple of
...ing element of another row (or column)
... value of the determinant remain unchanged.

$$\begin{vmatrix} a_1+2a_2 & b_1+2b_2 & c_1+2c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0.$$

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$C_2 \leftarrow C_2 - C_1$
$C_3 \leftarrow C_3 - C_1$

$$= abc \begin{vmatrix} 1 & 1-1 & 1-1 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$= abc \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$= abc\,[(b-a)(c^2-a^2) - (c-a)(b^2-a^2)]$

$= abc\,[(b-a)(c-a)(c+a) - (c-a)(b-a)(b+a)]$

$= abc\,(b-a)(c-a)\,[c+a-(b+a)]$

$= abc\,(b-a)(c-a)\,[c+\cancel{a}-b-\cancel{a}]$

$= abc\,(b-a)(c-a)(c-b)$

$= abc\,(-(a-b))(c-a)(-(b-c))$

$= abc\,(c-a)(a-b)(b-c)$

$= abc\,(a-b)(b-c)(c-a)$

2016

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$$

Ans: $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix} = - \begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$

$C_1 \leftarrow C_2$

$R_1 \leftrightarrow R_2$

$$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} \quad \text{Ans:}$$

प्रश्न $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

L.H.S $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$R_1 \leftarrow R_1 + R_2 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$C_2 \leftarrow C_2 - C_1$

$C_3 \leftarrow C_3 - C_1$

$$= (a+b+c) \begin{vmatrix} 1 & 1-1 & 1-1 \\ 2b & (b-c-a)-2b & 2b-2b \\ 2c & 2c-2c & c-a-b-2c \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)\left[(-(a+b+c))(-(a+b+c)) - 0\right]$$

$$= (a+b+c) \cdot (a+b+c)^2$$

$$= (a+b+c)^3$$

2014

$$\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$$ Solving for x.

$R_1 \leftarrow R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} x+1+\omega+\omega^2 & \omega+x+\omega^2+1 & \omega^2+1+x+\omega \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$$

$$\Rightarrow (x+1+\omega+\omega^2) \begin{vmatrix} 1 & 1 & 1 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$$

$$\Rightarrow (x+0) \begin{vmatrix} 1 & 1 & 1 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0 \quad [1+\omega+\omega^2=0]$$

$C_2 \leftarrow C_2 - C_1$, $C_3 \leftarrow C_3 - C_1$

$$\Rightarrow x \begin{vmatrix} 1 & 1-1 & 1-1 \\ \omega & x+\omega^2-\omega & 1-\omega \\ \omega^2 & 1-\omega^2 & x+\omega-\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow x \begin{vmatrix} 1 & 0 & 0 \\ \omega & x+\omega^2-\omega & 1-\omega \\ \omega^2 & 1-\omega^2 & x+\omega-\omega^2 \end{vmatrix} = 0$$

$\Rightarrow x[1((x+\omega^2-\omega)\cdot(x+\omega-\omega^2) - (1-\omega^2)(1-\omega))] = 0$

$\Rightarrow x[x^2 + \cancel{x\omega} - \cancel{x\omega^2} + \cancel{\omega^2 x} + \omega^2 - \omega^4 - \cancel{\omega x} - \omega^2 + \omega^3 - 1 + \omega + \omega^2 - \omega^3] = 0$

$\Rightarrow x(x^2 + \cancel{1} - \cancel{\omega} - \cancel{\omega^2} - \cancel{1} + \cancel{\omega} + \cancel{\omega^2}) = 0$

$\Rightarrow x \cdot x^2 = 0$

$\Rightarrow x^3 = 0$. $\Rightarrow \boxed{x=0}$

Q
$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

Ans: L.H.S
$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

$$= abc \begin{vmatrix} \frac{b^2+c^2}{a} & b & c \\ a & \frac{c^2+a^2}{b} & c \\ a & b & \frac{a^2+b^2}{c} \end{vmatrix}$$

$$= \begin{vmatrix} b^2+c^2 & b^2 & c^2 \\ a^2 & c^2+a^2 & c^2 \\ a^2 & b^2 & a^2+b^2 \end{vmatrix}$$

$C_1 \leftarrow C_1 - C_2 - C_3$

$$= \begin{vmatrix} b^2+c^2-b^2-c^2 & b^2 & c^2 \\ a^2-c^2-a^2-c^2 & c^2+a^2 & c^2 \\ a^2-b^2-a^2-b^2 & b^2 & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b^2 & c^2 \\ -2c^2 & c^2+a^2 & c^2 \\ -2b^2 & b^2 & a^2+b^2 \end{vmatrix}$$

$$= (-2) \begin{vmatrix} 0 & b^2 & c^2 \\ c^2 & c^2+a^2 & c^2 \\ b^2 & b^2 & a^2+b^2 \end{vmatrix}$$

$= (-2) \left[(-b^2)\left[c^2(a^2+b^2) - b^2c^2\right] + (c^2)\left[c^2b^2 - b^2(c^2+a^2)\right] \right]$

$= (-2) \left\{ (-b^2)\left(c^2a^2 + \cancel{c^2b^2} - \cancel{b^2c^2}\right) + c^2\left(\cancel{c^2b^2} - \cancel{b^2c^2} - b^2a^2\right) \right\}$

$= (-2) \left[-b^2c^2a^2 - c^2b^2a^2 \right]$

$= (-2) \left\{ -a^2b^2c^2 - a^2b^2c^2 \right\}$

$= (-2) \left\{ -2a^2b^2c^2 \right\}$

$= 4a^2b^2c^2$

viii) $$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$

Ans: L.H.S $$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= abc \begin{vmatrix} \frac{b+c}{a} & 1 & 1 \\ 1 & \frac{c+a}{b} & 1 \\ 1 & 1 & \frac{a+b}{c} \end{vmatrix}$$

$$= \begin{vmatrix} b+c & b & c \\ a & c+a & c \\ a & b & a+b \end{vmatrix}$$

$C_1 \leftarrow C_1 - C_2 - C_3$

$$= \begin{vmatrix} b-c-b-c & b & c \\ a-c-a-c & c+a & c \\ a-b-a-b & b & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b & c \\ -2c & c+a & c \\ -2b & b & a+b \end{vmatrix}$$

$$= (-2) \begin{vmatrix} 0 & b & c \\ c & c+a & c \\ b & b & a+b \end{vmatrix}$$

$= (-2) \left[(-b)(ca+bc-cb) + c(cb-cb+ab)\right]$

$= (-2)\{-abc - abc\}$

$= (-2)(-2abc)$

$= 4abc$

Q ii) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

Ans: L.H.S $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$$abc \begin{vmatrix} \frac{1}{a}+1 & 1/b & 1/c \\ 1/a & 1/b+1 & 1/c \\ 1/a & 1/b & 1/c+1 \end{vmatrix}$$

$C_1 \leftarrow C_1 + C_2 + C_3$

$$= abc \begin{vmatrix} 1+\frac{1}{a}+1/b+1/c & 1/b & 1/c \\ 1+1/a+1/b+1/c & 1/b+1 & 1/c \\ 1+1/a+1/b+1/c & 1/b & 1/c+1 \end{vmatrix}$$

$$= \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right) abc \begin{vmatrix} 1 & 1/b & 1/c \\ 1 & 1/b+1 & 1/c \\ 1 & 1/b & 1/c+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+1/c\right) \begin{vmatrix} 1 & 1 & 1 \\ 1/b & 1/b+1 & 1/b \\ 1/c & 1/c & 1/c+1 \end{vmatrix}$$

$C_2 \leftarrow C_2 - C_1$

$C_3 \leftarrow C_3 - C_1$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+1/c\right) \begin{vmatrix} 1 & 0 & 0 \\ 1/b & 1 & 0 \\ 1/c & 0 & 1 \end{vmatrix}$$

$$= (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)((1)(1-0))$$

$$= (abc)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Cramer's Rule

Consider a system of linear equation

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The soln is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, $z = \frac{D_3}{D}$.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

Properties

① If $D \neq 0$, Then the system is consistent, unique solution.

② If $D = 0$, $D_1 = 0 = D_2 = D_3 = 0$
Consistent, infinite no. of solution

③ If $D = 0$ at least are D_1, D_2 and D_3 is not zero, then the inconsistent & no. solution

Solve by cramer's rule

$3x + 2y = 7$

$2x - 3y = -4$

$$D = \begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix} = -9 - 4 = -13 \neq 0$$

$$D_1 = \begin{vmatrix} 7 & 2 \\ -4 & -3 \end{vmatrix} = -21 + 8 = -13$$

$$D_2 = \begin{vmatrix} 3 & 7 \\ 2 & -4 \end{vmatrix} = -12 - 14 = -26$$

$$x = \frac{D_1}{D} = \frac{-13}{-13} = 1$$

$$y = \frac{D_2}{D} = \frac{-26}{-13} = 2$$

Matrix

order $m \times n$ $\left[\begin{matrix} m = \text{row} \\ n = \text{column} \end{matrix}\right]$

$$\begin{array}{c|c}
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} & \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \\
\begin{matrix} C_1 & C_2 & C_3 \end{matrix} &
\end{array}$$

Kinds of Matrix

① Raw matrix / Row vector

$(a_1 \; b_1)_{1\times 2}$ $\quad (a \; b \; c)_{1\times 3}$

② Column matrix / Column vector

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}_{2\times 1}$ $\quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{3\times 1}$

③ Square / Rectangular matrix

Square matrix — no of rows = no. of columns

$\begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$

Rectangular matrix — no of rows $\neq$ No of columns

$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 1 & 5 \end{pmatrix}_{2\times 3}$

In a square matrix

diagonal matrix $= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

scalar matrix $= \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$

Identity (unit) matrix

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Transpose of a matrix (A' or A^T)

The transpose of a matrix is obtained by changing the rows into corresponding columns

Transpose of a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Symmetric Matrix

$A = A^T$ $[a_{ij} = a_{ji}]$

$a_{ii} = k$.

Ex.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 5 \\ 1 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 5 \\ 1 & 5 & 6 \end{bmatrix}$$

Skew symmetric matrix

$A = -A^T$ $(a_{ij} = -a_{ji})$

$a_{ii} = 0$.

Ex.

$$A = \begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & -5 \\ -4 & 5 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$$

$$-A^T = \begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & -5 \\ -4 & 5 & 0 \end{bmatrix}$$

$A + A^T$ = Symmetric

$A - A^T$ = skew symmetric

$$A = \begin{bmatrix} 10 & 28 & 0 \\ 25 & -6 & 5 \\ 15 & 2 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 10 & 25 & 15 \\ 28 & -6 & 2 \\ 0 & 5 & 3 \end{bmatrix}$$

$$P = A + A^T = \begin{bmatrix} 20 & 53 & 15 \\ 53 & -12 & 7 \\ 15 & 7 & 6 \end{bmatrix}$$

$$Q = A - A^T \begin{bmatrix} 0 & 3 & -15 \\ -3 & 0 & 3 \\ 15 & -3 & 0 \end{bmatrix}$$

Q. Express $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ -1 & 5 & -2 \end{bmatrix}$ as the sum of a symmetric and skew symmetric matrix

Ans) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ -1 & 5 & -2 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 0 & 5 \\ 3 & 1 & -2 \end{bmatrix}$

$$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 2 & 6 & 2 \\ 6 & 0 & 6 \\ 2 & 6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & 3 \\ 1 & 3 & -2 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & 3 \\ 1 & 3 & -2 \end{bmatrix}$$

$\Rightarrow P = P^T$

Hence P is a symmetric matrix.

$$Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & -2 & 4 \\ 2 & 0 & -4 \\ -4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

$$Q^T = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

$$-Q^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

$\Rightarrow Q = -Q^T$

Hence Q is a skew symmetric matrix

$$P + Q = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & 3 \\ 1 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ -1 & 5 & -2 \end{bmatrix} = A$$

Hence matrix A is the sum of symmetric matrix and skew symmetric matrix.

Minor and co-factor

The minor of any element a_{ij} of a matrix is the value of the determinant obtained by omitting i^{th} & j^{th} of the matrix. It is denoted by M_{ij} similarly the co-factor of the element a_{ij} is denoted by C_{ij} and is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 0 \\ -3 & 1 \end{vmatrix} = 1(5-0) = 5$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1(1-0) = -1$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 5 \\ 2 & -3 \end{vmatrix} = 1(-3-10) = 1(-13) = -13$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = -1(-1+6) = -5$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = 1(3-4) = -1$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 3 & -1 \\ 2 & -3 \end{vmatrix} = -1(-9+2) = 7$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 2 \\ 5 & 0 \end{vmatrix} = 1(0-10) = -10$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} = -1(0-2) = 2$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 3 & -1 \\ 1 & 5 \end{vmatrix} = 1(15+1) = 16$$

Cofactor of matrix of $A = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$

$$\text{Adj } A = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 5 & -5 & -10 \\ -1 & -1 & 2 \\ -13 & 7 & 6 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -1 & 2 \\ 1 & 5 & 0 \\ 2 & -3 & 1 \end{vmatrix}$$

$= 3(5-0) + 1(1-0) + 2(-3-10)$

$= 15 + 1 - 26 = -10.$

$A \neq 0.$

A^{-1} exist.

$$\boxed{A^{-1} = \frac{\text{Adj } A}{|A|}}$$

$$A^{-1} = -\frac{1}{10}\begin{bmatrix} 5 & -5 & -10 \\ -1 & -1 & 2 \\ -3 & 7 & 16 \end{bmatrix}$$

<u>Algebra of matrix</u>

<u>Addition</u>

$(A)_{m\times n} + (B)_{m\times n} = (C)_{m\times n}$

<u>multiplication</u>

$(A)_{m\times n} \, (B)_{n\times p} = (C)_{m\times p}$.

$$\begin{array}{c} \\ \begin{pmatrix} 0 & 1 & 2 \\ 3 & -1 & 4 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \end{array} \\ (2\times 3) \end{array} \begin{array}{cc} C_1 & C_2 \\ \end{array} \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 4 \end{pmatrix}_{3\times 2} = \begin{pmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{pmatrix}_{2\times 2}$$

$$\begin{pmatrix} 0.2+1.1+2.0 & 0.3+1.0+2.4 \\ 3.2+(-1).1+4.0 & 3.3+(-1)\cdot 0+4.4 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 5 & 25 \end{pmatrix}$$

Solve in matrix method.

Consider a system of linear equation

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$b_3x + b_3y + c_3z = d_3$$

The matrix form of the above system of the linear equation is given by $AX = B$

where,

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = B$$
$$A^{-1}(AX) = A^{-1}B$$
$$(A^{-1}A)X = A^{-1}B$$
$$IX = A^{-1}B$$
$$\boxed{X = A^{-1}B} \Rightarrow \boxed{X = \frac{\text{Adj}A}{|A|} B} \quad A^{-1} = \frac{\text{Adj}A}{|A|}$$

Q. Solve by matrix method.

$$x + 2y = 3$$
$$3x + y = 4$$

The matrix form is given by

$$AX = B$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1-6 = -5 \neq 0$$

A^{-1} exist.

$C_{11} = (-1)^{1+1}.1 = 1$ $\quad C_{12} = (-1)^{1+2}.3 = -3$

$C_{21} = (-1)^{2+1}\,2 = -2$ $\quad C_{22} = (-1)^{2+2}.1 = 1$

$$\text{Adj } A = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}}{-5}$$

$$= -\frac{1}{5} \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

$$X = A^{-1} B = -\frac{1}{5} \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$X = -\frac{1}{5} \begin{bmatrix} 1 \times 3 - 2 \times 4 \\ -3 \times 3 + 1 \times 4 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 3 - 8 \\ -9 + 4 \end{bmatrix}$$

$$X = -\frac{1}{5} \begin{bmatrix} -5 \\ -5 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x = 1$$

$$y = 1$$

Q Solve the following linear algebric equation using inverse of a matrix.

$x + y + z = 4$

$2x - y + 3z = 1$

$3x + 2y - z = 1$

Ans:- Given algebric linear equations are

$x + y + z = 4$

$2x - y + 3z = 1$

$3x + 2y - z = 1$

The matrix form is given by $AX = B$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 1(1-6) - 1(-2-9) + 1(4+3)$$

$$= -5 + 11 + 7 = -13 \neq 0$$

A^{-1} is exist.

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} = 1(1-6) = -5$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -1(-2-9) = 11$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 1(4+3) = 7$$

$$C_{21} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1(-1-2) = 3$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = 1(-1-3 = -4$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -1(2-3) = 1$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 1(3+1) = 4$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 1(-1-2) = -3$$

$$\operatorname{Adj} A = \begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{\operatorname{Adj} A}{A} = \frac{\begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & 3 \end{bmatrix}}{13} = \frac{1}{13}\begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & 3 \end{bmatrix}$$

$$X = A^{-1} \cdot B = \frac{1}{13}\begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -20+3+4 \\ 44-4-1 \\ 28+1-3 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -13 \\ 39 \\ 26 \end{bmatrix}.$$

$$X = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

$\Rightarrow x = -1, \quad y = 3, \quad z = 2$