MATRIX

Minor

The minor of an element a_{ij} of a matrix is the value of the determinant obtained by deleting i^{th} row & j^{th} column of the matrix.

Rank of a Matrix

The maximal order of non-zero minor of a matrix is called rank of a Matrix.

Equivalently: A matrix have rank 'r' if

(1) Their exist atleast one non-zero minor of order r.

(2) All the minor of order more than 'r' vanishes.

Ex: Find the rank of the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

Ans Minor of order - 2

$$\begin{vmatrix} 3 & -1 \\ -6 & 2 \end{vmatrix} = 6-6 = 0$$

next minor of order - 2 $\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} = -4-4 = -8 \neq 0$

Minor of order - 3

$$\begin{vmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{vmatrix} = -3 \begin{vmatrix} -1 & -1 & 2 \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{vmatrix} = -3.0 = 0$$

Conclusion

Minor of order $2 \neq 0$ (non-zero)

Minor of order $3 = 0$

$\Rightarrow$ Rank of the matrix is 2.

* We must try to search non-zero minor of order 2/3. (If exist)

Elementary ~~Row~~ Transformation of a Matrix

Elementary transformation can be done using only rows or only Columns.

For Rows / Columns

(1) The interchange any two rows (Columns)

(2) The multiplication of any row (Column) by a non-zero Scalar.

(3). The addition/substraction of a Constant ($\neq 0$) multiple of the elements of any row (Column) to the corresponding elements of any other row (Column).

Equivalent matrix. It is a matrix obtained by a sequence of elementary transformations of a Matrix.

* Two equivalent matrix have same order and rank.

Consistency of linear system of equations

Consider a System of linear equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Let.

$$A = \text{Coefficient matrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

k = Augmented matrix = $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$

Let

Rank of Coefficient matrix = r

Rank of Augmented matrix = r'

No of Variable = 'n' here. '3'

(1) If $r = r' = n \Rightarrow$ The equations are Consistent and have Unique Solution.

(2) If $r = r' < n \Rightarrow$ The equations are Consistent and have Infinite number of Solutions

(3) If $r \neq r''$ $\Rightarrow$ The equations are Inconsistent. (no-Solutions)

Ex Investigate the values of λ & b So that the equations $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = b$ have

(i) no Solution (ii) Unique Solution (iii) Infinite no of Solution

Soln

We have $\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ b \end{bmatrix}$

Minor of order-3 of Coefficient matrix

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5-\lambda)$$

1 Unique Solution

The system admits of Unique Solution iff the rank of the coefficient matrix is 3

$\Rightarrow 15(5-\lambda) \neq 0$

$\Rightarrow \lambda \neq 5$

So For $\lambda \neq 5$ and any values of b, the system posses Unique Solution.

Infinite no of Solution

In this case we have to make rank of both coefficient matrix & augmented matrix be '2' < 3.

So Rank of the coefficient matrix will be 2 if $\lambda - 5 = 0 \Rightarrow \boxed{\lambda = 5}$

Augmented matrix
$$\begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \cancel{\lambda} & b \end{bmatrix}$$

For $\lambda = 5$ and if $b = 9$, R_1 & R_3 are equal

$\Rightarrow$ All the minor of order '3' vanishes (zero)

$\Rightarrow$ Rank of Augmented matrix $= 2$.

$\Rightarrow$ For $\lambda = 5, b = 9$; the system posses infinite no. of Solution.

No Solution

Rank of coefficient matrix $= 2 \neq$ Rank of Augmented matrix $= 3$

So For $\lambda = 5$, $b \neq 9$, the system posses no solution

Linear Differential Equations

Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

A differential equation of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = 0$$

is called a linear homogeneous differential equation with constant coefficients.

Similarly A differential equation of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = g(x)$$

is called a linear non-homogeneous differential equation with constant coefficients.

$k_1, k_2, \ldots k_n$ are constants.

Solution of Linear homogeneous diff eqn.

Using $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$, $\ldots$ $\frac{d^n}{dx^n} = D^n$ the diff eqn can be written in symbolic form as $f(D)\, y = 0$

Considering $f(D) = 0$ which is called auxillary Equation.

Solve the A.E.

$$f(D) = 0$$

Let the solutions are $m_1, m_2, m_3, \ldots, m_n$

Case - I distinct roots

$$m_1 \neq m_2 \neq m_3 \neq \ldots \neq m_n$$

General solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

Case - II Equal roots

(a) $m_1 = m_2 \neq m_3 \neq m_4$

$$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots$$

(b) $m_1 = m_2 = m_3 \neq m_4 \neq \ldots$

$$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \ldots$$

(c) $m_1 = m_2 \neq m_3 = m_4 \neq m_5 \neq \ldots$

$$y = (c_1 + c_2 x) e^{m_1 x} + (c_3 + c_4 x) e^{m_3 x} + \ldots$$

Case - III Imaginary roots

$$m = \alpha + i\beta.$$

$$y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$$

* If imaginary roots are equal.

then $y = e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$

Example Solve $(D^2-4D+4)y=0$

A.E. $D^2-4D+4=0$

$(D-2)^2=0 \Rightarrow D=2,2$ Equal roots

G.S. $y=(C_1+C_2x)e^{2x}$

Example Solve $(D^3+D^2+4D+4)y=0$

A.E. $D^3+D^2+4D+4=0$

$(D^2+4)(D+1)=0$

for $D+1=0$ for $D^2+4=0$

$\boxed{D=-1}$ $D^2=-4$

$D=\sqrt{-4}=0\pm 2i$

$y=c_1e^{-x}+e^{0x}[c_2\cos 2x+c_3\sin 2x]$

$y=c_1e^{-x}+c_2\cos 2x+c_3\sin 2x$ Ans.

Solution of Linear Non-homogeneous equation

Symbolic form $f(D)y=g(x)$.

The general solution of a linear non-homogeneous diff eq^n is $y=C.F.+P.I.$

C.F = Complementary function.

P.I. = Particular Integral.

C.F can be found by solving the diff. equation $f(D)y=0$ [Solution of L.homogeneous diff eq^n]

Rules for finding Particular Integral

Case - I If $g(x) = e^{ax}$

$$f(D)y = g(x) = e^{ax}$$

P.I. $\dfrac{1}{f(D)} e^{ax} = \dfrac{e^{ax}}{f(a)}$, $f(a) \neq 0$

If $f(a) = 0$ then

$$\frac{1}{f(D)} \cdot e^{ax} = \frac{x \cdot e^{ax}}{f'(a)}, \quad f'(a) \neq 0$$

If $f'(a) = 0$ then

$$\frac{1}{f(D)} \cdot e^{ax} = \frac{x^2 e^{ax}}{f''(a)}, \quad f''(a) \neq 0$$

and so on.

Case - II

If $g(x) = \sin(ax+b) / \cos(ax+b)$

$$f(D^2)\, y = \sin(ax+b)$$

P.I. $\dfrac{1}{f(D^2)} \cdot \sin(ax+b) = \begin{cases} \dfrac{\sin(ax+b)}{f(-a^2)}, & f(-a^2) \neq 0 \\ \dfrac{x \cdot \sin(ax+b)}{f'(-a^2)}, & f'(-a^2) \neq 0 \end{cases}$

If $f(-a^2) = 0$ then (second case)

and so on.

Case - III

If $g(x) = x^m$ (a polynomial)

$$f(D)\,y = x^m$$

$$\text{P.I.} \quad \frac{1}{f(D)}\cdot x^m = [f(D)]^{-1}\, x^m$$

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Binomial Th^m.

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$$

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$$

$$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$$

$$(1+D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$$

Case - IV

If $g(x) = e^{ax}\cdot V(x)$ [$V(x)$ is any function]

$$f(D)\,y = e^{ax}\,V(x)$$

$$\text{P.I.} \quad \frac{1}{f(D)}\, e^{ax}\cdot V(x) = e^{ax}\cdot \frac{1}{f(D+a)}\, V(x).$$

Case - V

$$\frac{1}{D-a}\, g(x) = e^{ax}\int g(x)\, e^{-ax}\, dx.$$

Linear differential Equation with Variable Coefficient.

① Cauchy Equation.

$$x^n k_1 \frac{d^n y}{dx^n} + k_2 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_{n-1} x \frac{dy}{dx} + k_n y = g(x)$$

Solving Technique

Put $x = e^t \Rightarrow t = \log x \Rightarrow \frac{dt}{dx} = \frac{1}{x}$

Now $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$

$\Rightarrow x \frac{dy}{dx} = \frac{dy}{dt} = D'y$ where $D' = \frac{d}{dt}$

Similarly $x^2 \frac{d^2y}{dx^2} = D'(D'-1)y$

$x^3 \frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y$

~~Putting~~ Using the above substitution, the variable coeff linear diff eqn will converted to a linear diff. eqn with constant coefficient form.

② Legendre's Equation

$$k_1 (ax+b)^n \frac{d^n y}{dx^n} + k_2 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_n y = g(x)$$

Here Put $(ax+b) = e^t \Rightarrow t = \log(ax+b)$

$\frac{dt}{dx} = \frac{a}{ax+b}$.

Now

$(ax+b)\frac{dy}{dx} = a \cdot \frac{dy}{dt} = a \cdot D'y$, $D' = \frac{d}{dt}$

$(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D'(D'-1) y$ and so on.

Partial differential Equations

A partial differential Equations is that in which there are two or more independent variables and partial differential coefficients with respect to any of them.

Consider a P. differential Equation

$$z = f(x, y).$$

We shall employ the following notation:

$$\frac{\partial z}{\partial x} = p \quad , \quad \frac{\partial z}{\partial y} = q$$

$$\frac{\partial^2 z}{\partial x^2} = r \quad , \quad \frac{\partial^2 z}{\partial x \partial y} = s \quad , \quad \frac{\partial^2 z}{\partial y^2} = t .$$

Formation of Partial differential Equation

① By eliminating arbitary Constants.

② By Eliminating arbitary functions.

Ex Form a P.d.E. from the equation $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad —(1)$$

$$2 \cdot \frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow p = \frac{x}{a^2} \Rightarrow \frac{p}{x} = \frac{1}{a^2} \quad —(2)$$

$$2 \cdot \frac{\partial z}{\partial y} = \frac{2y}{b^2} \Rightarrow q = \frac{y}{b^2} = \frac{q}{y} = \frac{1}{b^2} \quad —(3)$$

By Putting the values of $\frac{1}{a^2}$ & $\frac{1}{b^2}$ in eqn (1)

We have

$$2z = x^2 \cdot \frac{p}{x} + y^2 \cdot \frac{q}{y}$$

$$\Rightarrow \boxed{2z = px + qy}$$

Example

Form a P.D.E from $z = f(x^2-y^2)$

$$z = f(x^2-y^2)$$

$$p = \frac{\partial z}{\partial x} = f'(x^2-y^2)\cdot 2x$$

$$q = \frac{\partial z}{\partial y} = f'(x^2-y^2)(-2y)$$

Now $\frac{p}{q} = \frac{f'(x^2-y^2)\cdot 2x}{f'(x^2-y^2)(-2y)}$

$\Rightarrow \frac{p}{q} = \frac{x}{-y} \Rightarrow -py = qx$

$\Rightarrow \boxed{qx + py = 0}$

Linear Equations of the first order

A linear P.D.E of the first order is commonly known as Lagrange's linear Eqn of the form $P(x)\,p + Q(y)\cdot q = R(z)$.

To solve the equation.

(1) Form the subsidiary Equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(2) Solve the subsidiary Equation by
(i) direction integration (ii) taking multipliers.

(3) Write of complete solution
as $f(u,v) = 0$ or $u = f(v)$

Where u, v are solutions of subsidiary Equation sty@gmail -

Example: Solve the P.D.E.

$$x^2(y-z)p + y^2(z-x)q = z^2(x-y)$$

Sol: Subsidiary Equations

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

By taking multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

Each fraction $\dfrac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{xy - xz + yz - yx + zx - yz}$

$\Rightarrow \dfrac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0} \Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

On Integration $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = \int 0$

$\Rightarrow \log x + \log y + \log z = \log c \Rightarrow \log xyz = \log c$

$\Rightarrow \boxed{xyz = c}$

By taking another multipliers $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$

Each fraction $\dfrac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{y - z + z - x + x - y}$

$\Rightarrow \dfrac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{0} \Rightarrow \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$

On Integration $\int \frac{dx}{x^2} + \int \frac{dy}{y^2} + \int \frac{dz}{z^2} = \int 0$

$\Rightarrow -\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c_2 \Rightarrow \boxed{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -c_2}$

Complete solution is $f\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$

FOURIER SERIES

Periodic function:

A function $f(x)$ is said to be of period 'c' if $f(x+c) = f(x), \forall x$

Ex. $\sin x$ and $\cos x$ are of period 2π.

$$f(x+2\pi) = \sin(x+2\pi) = \sin x$$

and. $$\cos(x+2\pi) = \cos x.$$

Result

1. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \cos nx \, dx = \left[\frac{\sin nx}{n}\right]_{\alpha}^{\alpha+2\pi} = 0$ $[n \neq 0$

2. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \sin nx \, dx = \left[-\frac{\cos nx}{n}\right]_{\alpha}^{\alpha+2\pi} = 0$ $[n \neq 0$

3. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \cos^2 nx \, dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} (1+\cos 2nx)\, dx$ $(n \neq 0)$

$$= \frac{1}{2}\left[\int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \cos 2nx \, dx\right]$$

$$= \frac{1}{2}\left[x\right]_{\alpha}^{\alpha+2\pi} + 0 = \frac{1}{2}\left[\alpha+2\pi-\alpha\right]$$

$$= \frac{1}{2}\cdot 2\pi = \pi$$

4. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \sin^2 nx \, dx = 0$ $(n \neq 0.)$

5. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \sin mx . \cos nx \, dx = 0$ $m \neq n$

6. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \sin mx . \sin nx \, dx = 0$ $m \neq n$

7. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \cos mx . \cos nx \, dx = 0$ $m \neq n.$

Dirichlet Conditions

1. $f(x)$ is Periodic, single valued and finite
2. $f(x)$ has finite number of discontinuities in any one Period.
3. $f(x)$ has at the most a finite number of maxima and minima.

Fourier Series

Let $f(x)$ be a Periodic function defined on $\alpha < x < \alpha + 2\pi$. Then its Fourier Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n, b_n are called Fourier Coefficients and.

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) . \cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) . \sin nx\, dx .$$

a_0, a_n, b_n are also called Euler's Coefficients.

*** Function having Point of discontinuity

$$f(x) = \begin{cases} g(x), & \alpha < x < c \\ h(x), & c < x < \alpha + 2\pi \end{cases}$$

'c' is the Point of discontinuity.

Then

$$a_0 = \frac{1}{\pi}\left[\int_{\alpha}^{c} g(x)\,dx + \int_{c}^{\alpha+2\pi} h(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} g(x).\cos nx\,dx + \int_{c}^{\alpha+2\pi} h(x).\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_{\alpha}^{c} g(x).\sin nx\,dx + \int_{c}^{\alpha+2\pi} h(x).\sin nx\,dx\right].$$

At the point of finite discontinuity $x = c$

$$f(x) = \frac{1}{2}\left[f(c-0) + f(c+0)\right].$$

Example:- Obtain the Fourier Series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$

Soln:- Let

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—(i)}$$

Then $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}dx = \frac{1}{\pi}\left|-e^{-x}\right|_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$

and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$

$= \frac{1}{\pi(n^2+1)}\left|e^{-x}(-\cos nx + n\sin nx)\right|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{n^2+1}$

$\therefore\; a_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2},\; a_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{5}$ etc.

Finally, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$

$$= \frac{1}{\pi(n^2+1)}\left| e^{-x}(-\sin nx - n\cos nx)\right|_0^{2\pi}$$

$$= \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{n}{n^2+1}$$

$\therefore\ b_1 = \frac{1-e^{-2\pi}}{\pi}\cdot\frac{1}{2},\quad b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{2}{5}$ etc.

Substituting the values of a_0, a_n, b_n in (i), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2}+\left(\frac{1}{2}\cos x+\frac{1}{5}\cos 2x+\frac{1}{10}\cos 3x+\cdots\right) + \left(\frac{1}{2}\sin x+\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x+\cdots\right)\right\}.$$

Example: Find the Fourier Series expansion for $f(x)$,

if $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi. \end{cases}$

Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$.

Soln: Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$ ——— (i)

Then $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_0^{\pi} x\,dx\right]$

$$= \frac{1}{\pi}\left[-\pi|x|_{-\pi}^{0} + |x^2/2|_0^{\pi}\right] = \frac{1}{\pi}(-\pi^2+\pi^2/2) = -\pi/2;$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0+\frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1)$$

$\therefore\ a_1 = \frac{-2}{\pi\cdot 1^2},\ a_2 = 0,\ a_3 = -\frac{2}{\pi\cdot 3^2},\ a_4 = 0,\ a_5 = -\frac{2}{\pi\cdot 5^2}$ etc.

Finally $b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$

$$= \frac{1}{\pi}\left[\left|\frac{\pi\cos nx}{n}\right|_{-\pi}^{0} + \left|-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right]$$

$$= \frac{1}{n}(1-2\cos n\pi)$$

$\therefore b_1 = 3,\ b_2 = -\frac{1}{2},\ b_3 = 1,\ b_4 = -\frac{1}{4}$ etc

Hence Substituting the values of a's & b's in (i), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots \quad \text{(ii)}$$

Which is the required result.

Putting $x=0$ in (ii), we obtain $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty\right)$ — (iii)

Now $f(x)$ is discontinuous at $x=0$. As a matter of fact

$f(0-0) = -\pi$ and $f(0+0) = 0$

$\therefore f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\pi/2$

Hence (iii) takes the form $-\pi/2 = -\pi/4 - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$

whence follows the result

Change of Interval

Normally we can represent periodic functions of period 2π by Fourier Series, but in many engineering problems, it is required to expand in other arbitrary period say '$2c$', by suitable substitution.

Put $z = \frac{\pi x}{c}$ $\Rightarrow x = \frac{cz}{\pi}$

$x = \alpha$ $\Rightarrow z = \frac{\pi\alpha}{c}$

$x = \alpha + 2c$ $\Rightarrow z = \frac{\pi(\alpha+2c)}{c} = \frac{\pi\alpha}{c} + 2\pi$

$f(x)$ is of period '$2c$' in $(\alpha, \alpha+2c)$ but

$\therefore f\left(\frac{cz}{\pi}\right) = F(z)$ is of period 2π in $\left(\frac{\pi\alpha}{c}, \frac{\pi\alpha}{c} + 2\pi\right)$

Hence the Fourier Series can be expanded in $(\alpha, \alpha+2c)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where $a_0 = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x) \cdot \cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x) \cdot \sin\frac{n\pi x}{c}\,dx.$$